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K-REGULARITY, cdh-FIBRANT HOCHSCHILD HOMOLOGY, AND A CONJECTURE OF VORST

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ABSTRACT. In this paper we prove that for an affine scheme essentially of finite type over a field F and of dimension d, K_{d+1}-regularity implies regularity, assuming that the characteristic of F is zero. This verifies a conjecture of Vorst.

INTRODUCTION

It is a well-known fact that algebraic K-theory is homotopy invariant as a functor on regular schemes; if X is a regular scheme then the natural map $K_n(X) \to K_n(X \times \mathbb{A}^1)$ is an isomorphism for all $n \in \mathbb{Z}$. This is false in general for nonregular schemes and rings.

To express this failure, Bass introduced the terminology that, for any contravariant functor $\mathcal{P}$ defined on schemes, a scheme X is called *$\mathcal{P}$-regular* if the pullback maps $\mathcal{P}(X) \to \mathcal{P}(X \times \mathbb{A}^r)$ are isomorphisms for all $r \geq 0$. If $X = \mathrm{Spec}(R)$, we also say that R is $\mathcal{P}$-regular. Thus regular schemes are K_n-regular for every n. In contrast, it was observed as long ago as [2] that a nonreduced affine scheme can never be K_1-regular. In particular, if A is an Artinian ring (that is, a 0-dimensional Noetherian ring) then A is regular (that is, reduced) if and only if A is K_1-regular. In [15], Vorst conjectured that for an affine scheme X, of finite type over a field F and of dimension d, regularity and K_{d+1}-regularity are equivalent; Vorst proved this conjecture for $d = 1$ (by proving that K_2-regularity implies normality).

In this paper, we prove Vorst's conjecture in all dimensions provided the characteristic of the ground field F is zero. In fact we prove a stronger statement. We say that X is *regular in codimension* $< n$ if $\mathrm{Sing}(X)$ has codimension $\geq n$ in X.

Let $\mathcal{F}_K$ denote the presheaf of spectra such that $\mathcal{F}_K(X)$ is the homotopy fiber of the natural map $K(X) \to KH(X)$, where $K(X)$ is the algebraic K-theory spectrum of X and $KH(X)$ is the homotopy K-theory of X defined in [16]. We write $\mathcal{F}_K(R)$ for $\mathcal{F}_K(\mathrm{Spec}(R))$.

Theorem 0.1. *Let R be a commutative ring which is essentially of finite type over a field F of characteristic 0. Then:*
(a) If $\mathcal{F}_K(R)$ is n-connected, then R is regular in codimension $< n$.
(b) If R is K_n-regular, then R is regular in codimension $< n$.
(c) (Vorst's conjecture) If R is $K_{1+\dim(R)}$-regular, then R is regular.

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It was observed in [16] that if X is K_n-regular then $K_i(X) \to KH_i(X)$ is an isomorphism for $i \leq n$, and a surjection for $i = n+1$, so that $\mathcal{F}_K(X)$ is n-connected. Thus (a) implies (b) in this theorem, and (c) is a special case of (b).

The bounds in (a) and (b) are the best possible, because it follows from Vorst's results ([15, Thm. A], [14, Thm. 3.6]) that for an affine singular seminormal curve X, $\mathcal{F}_K(X)$ is 1-connected, but not 2-connected. The converse of (c) is trivial, but those of (a) and (b) are false. Indeed, affine normal surfaces are regular in codimension 1 but may not be K_{-1}-regular, much less K_2-regular; see [20, 5.8.1].

Finally the analogue of (c) –and thus also of (a) and (b)– for K-theory of general nonaffine schemes is false. Indeed we give the following example of a nonreduced (and in particular nonregular) projective curve which is K_n-regular for all n.

Theorem 0.2. *Let (X, Q) be an elliptic curve over a field of characteristic 0, and P a rational point on X such that the line bundle $L = L(P - Q)$ does not have odd order in the Picard group $Pic(X)$. Write Y for the nonreduced scheme with the same underlying space as X but with structure sheaf $\mathcal{O}_Y = \mathcal{O} \oplus L$, where L is regarded as a square-zero ideal.*
Then Y is K_n-regular for all n, and $\mathcal{F}_K(Y)$ is contractible.

The proof of Theorem 0.1 employs results from our paper with M. Schlichting [4] that allow us to describe $\mathcal{F}_K$ in terms of cyclic homology; the necessary statements will be recalled in Section 1. In Section 2, we study the *cdh*-fibrant version of Hochschild homology and its Hodge decomposition. Section 3 contains a smoothness criterion (Theorem 3.1) using *cdh*-fibrant Hochschild homology, which is of independent interest and generalized in Theorem 4.11. The proof of part (a) of Theorem 0.1 is contained in Section 4 (see Theorem 4.12). As explained above, parts (b) and (c) follow from this. Finally Section 5 is devoted to the counterexample stated in Theorem 0.2 (and restated as Theorem 5.2).

Notation

All rings considered in this paper are commutative and noetherian. We shall write Sch/F for the category of schemes essentially of finite type over a field F. Objects of Sch/F shall be called *F-schemes*.

The category of spectra we use in this paper will not be critical. In order to minimize technical issues, we will use the terminology that a *spectrum* $\mathcal{E}$ is a sequence of simplicial sets $\mathcal{E}_n$ together with bonding maps $b_n : \mathcal{E}_n \to \Omega\mathcal{E}_{n+1}$. We say that $\mathcal{E}$ is an *Ω-spectrum* if all bonding maps are weak equivalences. A map of spectra is a strict map. We will use the model structure on the category of spectra defined in [3]. Note that in this model structure, every fibrant spectrum is an Ω-spectrum.

We say that a presheaf E of spectra on Sch/F satisfies the *Mayer-Vietoris-property* (or MV-property, for short) for a cartesian square of schemes

$$(\square) \qquad \begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

if applying E to this square results in a homotopy cartesian square of spectra. We say that E satisfies the Mayer-Vietoris property for a class of squares provided it satisfies the MV-property for each square in the class.

We say that E satisfies *Nisnevich descent* for Sch/F if E satisfies the MV-property for all elementary Nisnevich squares in Sch/F; an *elementary Nisnevich*

square is a cartesian square of schemes ($\Box$) for which $Y \to X$ is an open embedding, $X' \to X$ is étale and $(X' - Y') \to (X - Y)$ is an isomorphism. By [12, 4.4], this is equivalent to the assertion that $E(X) \to \mathbb{H}_{nis}(X, E)$ is a weak equivalence for each scheme X, where $\mathbb{H}_{nis}(-, E)$ is a fibrant replacement for E in a suitable model structure.

We say that E satisfies *cdh-descent* for Sch/F if E satisfies the MV-property for all elementary Nisnevich squares (Nisnevich descent) and for all abstract blow-up squares in Sch/F; an *abstract blow-up square* is a square ($\Box$) such that $Y \to X$ is a closed embedding, $X' \to X$ is proper and the induced morphism $(X' - Y')_{red} \to (X - Y)_{red}$ is an isomorphism. With M. Schlichting, we showed in Theorem 3.7 of [4] that *cdh*-descent is equivalent to the assertion that $E(X) \to \mathbb{H}_{cdh}(X, E)$ is a weak equivalence for each scheme X, where $\mathbb{H}_{cdh}(-, E)$ is a fibrant replacement for the presheaf E in a suitable model structure. We abbreviate $\mathbb{H}_{cdh}(-E)$ as $\mathbb{H}(-, E)$ if no confusion can arise, and write $\mathbb{H}_n(X, E) = \mathbb{H}^{-n}(X, E)$ for $\pi_n \mathbb{H}(X, E)$.

We use cohomological indexing for all chain complexes in this paper; for a complex A, $A[p]^q = A^{p+q}$. It is well-known that there is an Eilenberg-Mac Lane functor $A \mapsto |A|$ from chain complexes of abelian groups to spectra, and from presheaves of chain complexes of abelian groups to presheaves of spectra. This functor sends quasi-isomorphisms of complexes to weak homotopy equivalences of spectra, and satisfies $\pi_n(|A|) = H^{-n}(A)$. In this spirit, we will use descent terminology for presheaves of complexes.

For example, the Hochschild, cyclic, periodic and negative cyclic homology of schemes over a field k (such as F-schemes over a field $F \supseteq k$) can be defined using the Zariski hypercohomology of certain presheaves of complexes; see [18] and [4, 2.7] for precise definitions. We shall write these presheaves as $HH(/k)$, $HC(/k)$, $HP(/k)$ and $HN(/k)$, respectively and regard them as presheaves of either cochain complexes or spectra. When k is omitted, it is understood that $k = \mathbb{Q}$ is intended. Finally, we write $\Omega^i_{/k}$ for the presheaf $X \mapsto \Omega^i_{X/k}$, while $\Omega^\bullet_{/k}$ denotes the presheaf of algebraic de Rham complexes and $\Omega^{\le i}_{/k}$ denotes its brutal truncation in degree i.

1. *cdh*-DESCENT

In this section we recall the main results from [4], and prove that the failure of K-theory to be homotopy invariant can be measured using cyclic homology. We work on the category Sch/F of F-schemes essentially of finite type over a field F of characteristic 0.

Here are two of the main results of [4]. Recall that infinitesimal K-theory $\mathcal{K}^{inf}(X)$ is the homotopy fiber of the Jones-Goodwillie Chern character $K(X) \to HN(X)$. The first one is Theorem 4.6 of [4]:

Theorem 1.1. *The presheaf of spectra $\mathcal{K}^{inf}$ satisfies cdh-descent.*

The second one is a slight modification of Corollary 3.13 of [4].

Theorem 1.2. *For each subfield $k \subseteq F$, the presheaf $HP(/k)$ satisfies cdh-descent on* Sch/F*. In particular, HP satisfies cdh-descent.*

Proof. For $k = F$, this is proven in [4, Corollary 3.13]. As in *loc. cit.,* the result for $k \subset F$ follows from [4, Theorem 3.12], once we verify that the hypotheses hold. But this follows from three observations: (1) the Cuntz-Quillen excision theorem holds over k (see [5, 5.3], noting that the condition that the base field be algebraically

closed is not needed, see [5, p. 3]); (2) Goodwillie's theorem that periodic cyclic homology is invariant under infinitesimal extension does not require rings of finite type over k (see [6, II.5.1] or [10, 4.1.15]); and (3) the results of [4, Section 2] also hold for (Hochschild, cyclic, periodic, negative) homology over k. □

Example 1.3. Consider the presheaf of spectra KH associated to Homotopy K-theory. The main theorem of [7] states that $\mathbb{H}_{cdh}(-,K) \simeq KH$, and that KH satisfies *cdh* descent. Thus the following definition of $\mathcal{F}_K$ is compatible with the definition of $\mathcal{F}_K$ given in the introduction.

Definition 1.4. For any presheaf of spectra E, we write $\mathcal{F}_E$ for the homotopy fiber of $E \to \mathbb{H}_{cdh}(-,E)$. If $\mathcal{F}_E(X)$ is n-connected for some scheme X (for all X in some subcategory of Sch/F), we say that E satisfies n-*cdh*-descent on X (resp., on the subcategory). Note that if E satisfies n-*cdh*-descent for all n on a subcategory, then E satisfies *cdh*-descent on that subcategory.

Since the fibrant replacement functor $\mathbb{H}_{cdh}$ preserves (objectwise) fibration sequences, it follows that $\mathcal{F}$ does too. (See the first paragraph of [4, Section 5].) We record this as the following observation.

Lemma 1.5. *Let $E_1 \to E_2 \to E_3$ be a fibration sequence of presheaves of spectra. Then there is a natural induced fibration sequence*

$$\mathcal{F}_{E_1} \to \mathcal{F}_{E_2} \to \mathcal{F}_{E_3}.$$

Theorem 1.6. *For any scheme X, essentially of finite type over a field of characteristic 0, the Chern character $K \to HN = HN(/\mathbb{Q})$ induces natural weak equivalences*

$$\mathcal{F}_K(X) \xrightarrow{\simeq} \mathcal{F}_{HN}(X) \xleftarrow{\simeq} \Omega^{-1}\mathcal{F}_{HC}(X).$$

Proof. The first weak equivalence follows from Lemma 1.5 and Theorem 1.1. The second weak equivalence follows from Lemma 1.5, Theorem 1.2 and the SBI fibration sequence $\Omega HP \to \Omega^{-1}HC \to HN \to HP$. □

Corollary 1.7. *Let $X \in \mathrm{Sch}/F$. Then K satisfies $(n+1)$-cdh-descent on X if and only if HH satisfies n-cdh-descent on X.*

Proof. Since $HH(X)$ and $\mathbb{H}(X,HH)$ are n-connected for $n < -\dim(X)$ by [18], this follows from Theorem 1.6 and the SBI sequence $\Omega^{-1}HC \to HH \to HC$. □

2. *cdh*-fibrant Hochschild and cyclic homology.

In this section we study the *cdh*-fibrant version of Hochschild homology and its Hodge decomposition, and establish some of their basic properties.

For legibility, we will write a for the natural morphism of sites from the *cdh*-site to the Zariski site on Sch/F. If A is a Zariski sheaf, its associated *cdh* sheaf will be written as A_{cdh} or a^*A. We will simplify notation and write $H^*_{cdh}(X,A)$ for the cohomology of A_{cdh}. In particular this applies to the sheaf $X \mapsto \Omega^i_{X/k}$ of Kähler i-differential forms ($i \geq 0$); $H^*_{cdh}(X,\Omega^i_{/k})$ is the cohomology of $a^*\Omega^i_{/k}$. If A is a complex of presheaves of abelian groups on Sch/F, then we write $\mathbb{H}_{cdh}(A)$ for a *cdh*-fibrant replacement of A, and $\mathbb{H}_{cdh}(X,A)$ for its complex of sections over X; the usual hypercohomology $\mathbb{H}^n_{cdh}(X,A)$ is just $H^n\mathbb{H}_{cdh}(X,A)$. For example, if A

is a presheaf, considered as a complex concentrated in degree zero, then $\mathbb{H}_{cdh}(A)$ is just an injective resolution of the *cdh*-sheafification A_{cdh}, and $\mathbb{H}^n_{cdh}(X,A)$ is the usual cohomology $H^n_{cdh}(X,A_{cdh})$ of A_{cdh}.

When A is an unbounded complex, such as a complex representing Hochschild homology, then $\mathbb{H}_{cdh}(A)$ may be constructed using product total complexes of flasque Cartan-Eilenberg resolutions. This works because the columns of the Cartan-Eilenberg double complex are locally cohomologically bounded by [13].

The *cdh* site is Noetherian (every covering has a finite subcovering), so *cdh* cohomology commutes with filtered direct limits of sheaves. A typical application of this fact is that if M is a sheaf of F-modules and V is a vector space then $H^n_{cdh}(X, V\otimes_F M)\cong V\otimes_F H^n_{cdh}(X,M)$. (See [1], Exp. VI, 2.11 and 5.2.)

The Hochschild and cyclic homology of schemes over a field k (such as F-schemes over a field $F\supseteq k$) can be defined using the Zariski hypercohomology of certain presheaves of mixed complexes; see [18] and [4, 2.7] for precise definitions. It was observed in [19, 3.0] that, because the mixed complexes already admit a Hodge decomposition, so do the complexes $HH(/k)$, $HC(/k)$, $HP(/k)$ and $HN(/k)$. Taking fibrant replacements for any Grothendieck topology respects such product decompositions; the following proposition records this for the *cdh*-topology.

Proposition 2.1. *Let X be a scheme over a field k of characteristic* 0 *(such as an F-scheme for $F\supseteq k$). Then the cdh-fibrant Hochschild, cyclic, negative cyclic and periodic cyclic homology of X over k admit natural Hodge decompositions.*
That is, if H denotes any of $HH(/k)$, $HC(/k)$, $HP(/k)$ or $HN(/k)$ then:

$$\mathbb{H}_{cdh}(X,H)\cong\prod\mathbb{H}_{cdh}(X,H^{(i)}).$$

Moreover, using the computations of these decompositions provided in [19, Theorem 3.3] and the fact that all F-schemes are locally smooth in the *cdh*-topology, it is possible to compute the Hodge decomposition explicitly in terms of the *cdh*-hypercohomology of the de Rham complex.

Theorem 2.2. *Let $k\subseteq F$ be a subfield. There are natural isomorphisms for every F-scheme X:*

$$\begin{aligned}
\pi_n\mathbb{H}_{cdh}(X,HH^{(i)}(/k)) &\cong H^{i-n}_{cdh}(X,\Omega^i_{/k});\\
\pi_n\mathbb{H}_{cdh}(X,HC^{(i)}(/k)) &\cong \mathbb{H}^{2i-n}_{cdh}(X,\Omega^{\le i}_{/k});\\
\pi_n\mathbb{H}_{cdh}(X,HN^{(i)}(/k)) &\cong \mathbb{H}^{2i-n}_{cdh}(X,\Omega^{\ge i}_{/k});\\
\pi_n\mathbb{H}_{cdh}(X,HP^{(i)}(/k)) &\cong \mathbb{H}^{2i-n}_{cdh}(X,\Omega^{\bullet}_{/k}).
\end{aligned}$$

Proof. Let $\mathbf{C}(X)$ denote the mixed complex computing the Hochschild and cyclic homology of X over k. The functor $\mathbf{C}: X\mapsto\mathbf{C}(X)$ is a presheaf of mixed complexes. By [17, 9.8.12], there is a Hodge decomposition $\mathbf{C}\cong\prod_i\mathbf{C}^{(i)}$ and a natural map of mixed complexes $e:\mathbf{C}\to(\Omega^*_{X/k},0,d)$ that sends the Hochschild chain complex $HH^{(i)}(X)=(\mathbf{C}^{(i)}(X),b)$ to $\Omega^i_{X/k}[i]$. As observed in [19], the induced map on Connes' double complexes sends $\mathcal{B}^{(i)}_{**}$ to $\Omega^{\le i}_{X/k}[2i]$. It suffices to prove that these are locally quasi-isomorphisms for the *cdh* topology. This boils down to showing that e induces a quasi-isomorphism $HH^{(i)}(R/k)\to\Omega^i_{R/k}[i]$ for every regular local F-algebra R. For $k=F$, this is the Hochschild-Kostant-Rosenberg theorem ([17, 9.4.7]). The general case follows from this and the fact that R is the union of

smooth k-algebras. (It also follows from the Kassel-Sledsjœ spectral sequence of [9, 4.3a], which we recall in 4.1 below.) $\square$

Lemma 2.3. *Let R be an F-algebra essentially of finite type, $k \subseteq F$ a subfield. Then for each n, $HH(/k)$ satisfies n-cdh descent on $X = \operatorname{Spec}(R)$ if and only if the following three conditions hold simultaneously:*

$$HH_m^{(q)}(R/k) = 0 \quad \text{if } 0 \le q < m \le n; \tag{2.3a}$$

$$\Omega^q_{R/k} \to H^0_{cdh}(X, \Omega^q_{/k}) \quad \text{is bijective if } q \le n \text{ and onto if } q = n+1; \tag{2.3b}$$

$$H^p_{cdh}(X, \Omega^q_{/k}) = 0 \quad \text{if } p > 0 \text{ and } 0 \le q \le p+n+1. \tag{2.3c}$$

Note that (2.3a) is vacuous if $n \le 0$, and (2.3b) is vacuous if $n \le -2$. In particular, $HH(/k)$ satisfies (-2)-cdh-descent just in case $H^p_{cdh}(X, \Omega^q_{/k}) = 0$ for all $p > q \ge 0$.

Proof. This follows easily from the Hodge decomposition and the isomorphisms

$$HH_q^{(q)}(R/k) \cong \Omega^q_{R/k}, \quad \text{and} \quad HH_m^{(q)}(R/k) = 0 \text{ for } q > m.$$

In more detail, we see from 2.1 and 2.2 that the maps $HH_m^{(q)}(R/k) \to H^{q-m}_{cdh}(X, \Omega^q_{/k})$ must be isomorphisms for $m \le n$ and onto for $m = n+1$. $\square$

On smooth schemes, all our functors are well-behaved. Recall from [13] that the *scdh* topology on Sm/F is the restriction of the *cdh* topology on Sch/F. Since every scheme is locally smooth, it follows that $\mathbb{H}_{scdh}(X, A)$ is just $\mathbb{H}_{cdh}(X, A)$ for every presheaf A. (See the argument of the first part of the proof of [4, 3.12].) If A satisfies *scdh*-descent then $A(X) \cong \mathbb{H}_{cdh}(X, A)$ for all smooth X.

Recall that k is a subfield of F.

Theorem 2.4. *Let H denote any of: $HH(/k)$, $HC(/k)$, $HN(/k)$ or $HP(/k)$, and let $H^{(i)}$ denote the i^{th} component in the Hodge decomposition of H.*
Then H and $H^{(i)}$ satisfy scdh-descent on Sm/F. In particular, if X is smooth over F, then $H^{(i)}(X) \cong \mathbb{H}_{cdh}(X, H^{(i)})$.

Proof. Since every smooth scheme over F is locally a filtered limit of smooth affine schemes over k, and H commutes with limits of affine schemes, we may assume that $k = F$. By [4, 3.9, 2.9, and 2.10], Hochschild, cyclic, negative and periodic cyclic homology (relative to F) all satisfy *scdh*-descent on Sm/F.

By Proposition 2.1, the quasi-isomorphisms $H(X) \cong \mathbb{H}_{cdh}(X, H) = \mathbb{H}_{scdh}(X, H)$ induce quasi-isomorphisms $H^{(i)}(X) \cong \mathbb{H}^{(i)}_{cdh}(X, H) = \mathbb{H}^{(i)}_{scdh}(X, H)$ for all i. $\square$

The special case $H^*_{Zar}(X, \mathcal{O}) \cong H^*_{cdh}(X, \mathcal{O})$ (for smooth X) of the following corollary was proven in [4, 6.3].

Corollary 2.5. *If X is smooth over F, then $H^p_{Zar}(X, \Omega^i_{/k}) \cong H^p_{cdh}(X, \Omega^i_{/k})$ for all p and i. In particular, $\Omega^i_{X/k} \cong H^0_{cdh}(X, \Omega^i_{/k})$.*

Proof. Consider the map $e^{(i)} : HH^{(i)} = (\mathbf{C}^{(i)}, b) \to \Omega^i_{/F}[i]$ of complexes of Zariski sheaves. By [19, 3.3], it is a quasi-isomorphism over every smooth scheme X over F, inducing $HH^{(i)}_{i-p}(X) \cong H^{p-i}_{Zar}(X, \mathbf{C}^{(i)}) \cong H^p_{Zar}(X, \Omega^i_{/k})$. The map $e^{(i)}$ remains a quasi-isomorphism after sheafifying for the *cdh* topology, so that $\mathbb{H}^{p-i}_{cdh}(X, HH^{(i)}) \cong H^{p-i}_{cdh}(X, \mathbf{C}^{(i)}) \cong H^p_{cdh}(X, \Omega^i_{/k})$. By Theorem 2.4, $HH^{(i)}_n(X) \cong \mathbb{H}^{-n}_{cdh}(X, HH^{(i)})$, whence the result. $\square$

The next result is proven by copying the proof of [4, 6.1], replacing $\mathcal{O}$ with $\Omega^i_{/k}$.

Proposition 2.6. *If X is a d-dimensional scheme, essentially of finite type over F, and $k \subseteq F$ is a subfield, then*

$$H^d_{Zar}(X, \Omega^i_{/k}) \to H^d_{cdh}(X, \Omega^i_{/k})$$

is surjective. In particular, if X is affine and $d > 0$ then $H^d_{cdh}(X, \Omega^i_{/k}) = 0$.

The following useful theorem is proven in [13, 12.1].

Theorem 2.7. *For every abstract blow-up square ($\square$), and for every complex of sheaves of abelian groups A, there is a long exact Mayer-Vietoris sequence:*

$$\cdots H^n_{cdh}(X, A) \to H^n_{cdh}(X', A) \oplus H^n_{cdh}(Y, A) \to H^n_{cdh}(Y', A) \to H^{n+1}_{cdh}(X, A) \cdots$$

Consider the change-of-topology morphism $a : (\mathrm{Sch}/F)_{cdh} \to (\mathrm{Sch}/F)_{Zar}$.

Lemma 2.8. *If a Zariski sheaf M on Sch/F is a quasi-coherent sheaf (resp., coherent sheaf) on each X_{Zar}, and M satisfies scdh-descent on Sm/F, then the cohomology sheaves $R^q a_*(a^* M)$ are also quasi-coherent (resp., coherent) on each X_{Zar}.*

If $X = \mathrm{Spec}(R)$ is affine, then $R^q a_*(a^* M)$ is the quasi-coherent sheaf associated to the R-module $H^q_{cdh}(X, M)$, and the natural map $M(X) \to H^0_{cdh}(X, M)$ is R-linear.

Proof. We proceed by induction on $\dim X$, the case $\dim(X) = 0$ being clear. Pick a smooth proper birational cdh cover X' of X, as in [13, 5.9] or [11, 12.23], and form the abstract blow-up square ($\square$). By Theorem 2.7, we get a triangle on X_{Zar}: $Ra_*(a^*M)|_X \to Ra_*(a^*M)|_{X' \coprod Y} \to Ra_*(a^*M)|_{Y'}$ As the latter two terms have quasi-coherent (resp., coherent) cohomology sheaves, by induction and *scdh*-descent on X', so does the first.

If X is affine, then $H^p_{Zar}(X, R^q a_* M) = 0$ for $p > 0$. Hence the Leray spectral sequence collapses to yield $H^q_{cdh}(X, M) = H^0_{Zar}(X, R^q a_*(a^*M))$. $\square$

Corollary 2.9. *Suppose that $X = \mathrm{Spec}(R)$ is affine. Then $U \mapsto \pi_n \mathcal{F}_{HH(/k)}(U)$ and $U \mapsto \pi_n \mathbb{H}_{cdh}(U, HH(/k))$ are quasi-coherent Zariski sheaves on X for all n.*

3. A criterion for smoothness.

In this section we present a local criterion for smoothness of schemes over a field F, in terms of the Hochschild homology and *cdh*-fibrant Hochschild homology of their local rings over F (see 3.1). As an application we prove Vorst's conjecture for algebras of finite type over $\mathbb{Q}$ and their localizations at maximal ideals (see 3.3).

A stronger global version of the following result shall be proved in Section 4 below (see Theorem 4.11).

Recall that F is a field of characteristic 0.

Theorem 3.1. *Let R be the local ring of a d-dimensional F-algebra of finite type at a maximal ideal. If $HH(/F)$ satisfies d-cdh-descent on R, then R is smooth over F.*

Proof. Recall that $\Omega^\bullet_{/F}$ denotes the de Rham complex, whose terms are the Zariski sheaves $\Omega^i_{/F}$, while $\Omega^{\leq i}_{/F}$ denotes its brutal truncation in degrees at most i. By 2.2 and 1.2, we have isomorphisms

$$HP^{(j)}_n(X/F) \xrightarrow{\cong} \mathbb{H}^{2j-n}_{cdh}(X, \Omega^\bullet_{/F})$$

for any $X \in \mathrm{Sch}/F$, and all n and j. Moreover, by the proof of 2.2, this isomorphism factors through a natural map $e : HP_n^{(j)}(X/F) \to \mathbb{H}^{2j-n}_{Zar}(X, \Omega^\bullet_{/F})$. Now specialize to the case $X = \operatorname{Spec} R$, where R is as in the theorem. Since every *cdh* cover of X has a d-dimensional refinement, we have

$$\mathbb{H}^*_{cdh}(X, \Omega^\bullet_{/F}) = \mathbb{H}^*_{cdh}(X, \Omega^{\leq d}_{/F}).$$

Moreover, Lemma 2.3 implies that the hypercohomology spectral sequence for $\mathbb{H}^*_{cdh}$ degenerates to yield an isomorphism

$$H^*(\Omega^{\leq d}_{R/F}, d) \to \mathbb{H}^*_{cdh}(X, \Omega^{\leq d}_{/F}).$$

The canonical map $S : HP_n^{(j)}(R/F) \to HC_{n-2}^{(j-1)}(R/F)$ fits into the commutative diagram

$$\begin{tikzcd}
HP_{d+2}^{(d+1)}(R/F) \arrow[r, "e"'] \arrow[d, "S"'] & H^d_{dR}(R/F) \arrow[r] \arrow[d, hook] & \mathbb{H}^d_{cdh}(X, \Omega^\bullet_{/F}) \arrow[d, "\wr"] \\
HC_d^{(d)}(R/F) \arrow[r, "\cong", "e"'] & \Omega^d_{R/F}/d\Omega^{d-1}_{R/F} \arrow[r, "\cong"] & \mathbb{H}^d_{cdh}(X, \Omega^{\leq d}_{/F}).
\end{tikzcd}$$

We have seen that the top composite, the right vertical and both bottom arrows are isomorphisms. It follows that the middle vertical inclusion is the identity map, i.e., that $d\Omega^d_{R/F} = 0$. On the other hand, $d\Omega^d_{R/F}$ generates $\Omega^{d+1}_{R/F}$ as an R-module; therefore we can infer that $\Omega^{d+1}_{R/F} = 0$. By Lemma 3.2 below, R is regular, and hence smooth over F. $\square$

Lemma 3.2. *Let F be any perfect field. Suppose R is the local ring of a d-dimensional F-algebra of finite type at a maximal ideal. If $\Omega^{d+1}_{R/F} = 0$, then R is regular.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of R. Since $L := R/\mathfrak{m}$ is smooth over F, the Second Fundamental Theorem [17, 9.3.5] shows that there is an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \to L \otimes_R \Omega^1_{R/F}$ sending x to dx. Consequently, there is a surjection from $\Omega^{d+1}_{R/F}$ onto $\Lambda^{d+1}_L(\mathfrak{m}/\mathfrak{m}^2)$, which is a nonzero vector space unless R is regular. $\square$

As an application, we can now verify Vorst's Conjecture for algebras of finite type over $\mathbb{Q}$ and their localizations at maximal ideals.

Theorem 3.3. *Let R be a d-dimensional $\mathbb{Q}$-algebra which is either of finite type over $\mathbb{Q}$, or a localization of a $\mathbb{Q}$-algebra of finite type at a maximal ideal.*

If R is K_{d+1}-regular, then R is regular.

Proof. First assume R is of finite type over $\mathbb{Q}$, and K_{d+1}-regular. To prove R is regular, we may replace R by its localization at a maximal ideal; these local rings are also K_{d+1}-regular, by Vorst's localization theorem [14, 1.90]. Thus we are reduced to proving the theorem in the local case.

As remarked in the introduction, if R is K_{d+1}-regular, then $\mathcal{F}_K(R)$ is $(d+1)$-connected (see [16]). By Corollary 1.7, $\mathcal{F}_{HH(/F)}(R)$ is d-connected. Now Theorem 3.1 applies to prove that R is smooth over $\mathbb{Q}$ and hence regular. $\square$

4. Vorst's conjecture.

In this section we will prove Theorem 0.1. Throughout, F will be a fixed field of characteristic zero, $k \subseteq F$ a subfield, R an F-algebra essentially of finite type, and $X = \mathrm{Spec}(R)$. Note that we write HH for $HH(/\mathbb{Q})$.

Lemma 4.1. *(Kassel-Sledsjæ, [9, 4.3a]) Let $k \subseteq F$ and $p \geq 1$ be fixed. Then there is a bounded second quadrant homological spectral sequence ($0 \leq i < p$, $j \geq 0$):*

$$ {}_pE^1_{-i,i+j} = \Omega^i_{F/k} \otimes_F HH^{(p-i)}_{p-i+j}(R/F) \Rightarrow HH^{(p)}_{p+j}(R/k)$$

Lemma 4.2. *Let $k \subseteq F$ and $p \geq 1$ be fixed. Then there is a spectral sequence:*

$$ {}^pE_1^{i,j} = \Omega^i_{F/k} \otimes_F H^{i+j}_{cdh}(X, \Omega^{p-i}_{/F}) \Rightarrow H^{i+j}_{cdh}(X, \Omega^p_{/k}).$$

Proof. Consider the sheaf of ideals $I := \ker(\Omega^*_{/k} \to \Omega^*_{/F})$. The I-adic filtration of $\Omega^*_{/k}$ induces a filtration $\mathcal{G} = \mathcal{G}(p)$ on $\Omega^p_{/k}$. If R is any F-algebra essentially of finite type, we have a natural surjection

$$\Omega^i_{F/k} \otimes_F \Omega^{p-i}_{R/F} \twoheadrightarrow \mathcal{G}^i(R)/\mathcal{G}^{i+1}(R), \tag{4.3}$$

which is an isomorphism if R is smooth. Thus the cdh-sheafification of (4.3) is an isomorphism. Since $\Omega^i_{F/k}$ is a vector space, the spectral sequence of the lemma is the one associated to the corresponding filtration of the *cdh* sheaf $a^*\Omega^p_{/k}$. □

Lemma 4.4. *Let $X = \mathrm{Spec}(R)$ be affine, and fix $n \geq 0$. Assume that*

$$\Omega^q_{R/k} \to H^0_{cdh}(X, \Omega^q_{/k}) \quad \text{is bijective if } q \leq n \text{ and onto if } q = n+1, \tag{4.4a}$$

$$H^1_{cdh}(X, \Omega^q_{/F}) = 0 \text{ if } q \leq n+1. \tag{4.4b}$$

Then $\Omega^q_{R/F} \to H^0_{cdh}(X, \Omega^q_{/F})$ is bijective if $q \leq n$, and onto if $q = n+1$.

Proof. By induction on q. If $q = 0$, there is nothing to prove. Fix $q > 0$, and consider the filtration $\mathcal{G}^i = \mathcal{G}^i(q)$, $0 \leq i \leq q$, considered in the proof of Lemma 4.2. We have a commutative diagram

$$\begin{array}{ccc} \Omega^i_{F/k} \otimes_F \Omega^{q-i}_{R/F} & \longrightarrow & \mathcal{G}^i(R)/\mathcal{G}^{i+1}(R) \\ \downarrow & & \downarrow \\ \Omega^i_{F/k} \otimes_F H^0_{cdh}(X, \Omega^{q-i}_{/F}) & \longrightarrow & H^0_{cdh}(X, \mathcal{G}^i/\mathcal{G}^{i+1}). \end{array} \tag{4.5}$$

The top arrow is surjective for all i, and an isomorphism for $i = 0$. The bottom arrow is an isomorphism by the proof of 4.2. By the inductive hypothesis, the left vertical arrow is an isomorphism for $0 < i$. It follows that the top arrow is an isomorphism for all $0 \leq i \leq q$, and that the arrow on the right is an isomorphism for $i > 0$. By (4.4b) we have an exact sequence:

$$\begin{array}{ccccccc} 0 \to H^0_{cdh}(X, \mathcal{G}^{i+1}) & \to & H^0_{cdh}(X, \mathcal{G}^i) & \to & \Omega^i_{F/k} \otimes_F H^0_{cdh}(X, \Omega^{q-i}_{/F}) \\ & & & & \downarrow \\ & 0 \leftarrow & H^1_{cdh}(X, \mathcal{G}^i) & \leftarrow & H^1_{cdh}(X, \mathcal{G}^{i+1}) \end{array}$$

Since $\mathcal{G}^{q+1} = 0$, we deduce, by descending induction on i, that for all $i > 0$,

$$H^1_{cdh}(X, \mathcal{G}^i) = 0 \tag{4.6}$$

Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G}^{i+1}(R) \arrow[r] \arrow[d] & \mathcal{G}^{i}(R) \arrow[r] \arrow[d] & \Omega^i_{F/k}\otimes\Omega^{q-i}_{R/F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0_{cdh}(X,\mathcal{G}^{i+1}) \arrow[r] & H^0_{cdh}(X,\mathcal{G}^{i}) \arrow[r] & \Omega^i_{F/k}\otimes H^0_{cdh}(X,\Omega^{q-i}_{/F}) \arrow[r] & 0
\end{tikzcd}$$

Using descending induction on i again, we obtain from this diagram that

$$\mathcal{G}^i(R)\cong H^0_{cdh}(X,\mathcal{G}^i) \qquad (i>0). \tag{4.7}$$

We have a map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G}^{1}(R) \arrow[r] \arrow[d] & \Omega^q_{R/k} \arrow[r] \arrow[d] & \Omega^q_{R/F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0_{cdh}(X,\mathcal{G}^{1}) \arrow[r] & H^0_{cdh}(X,\Omega^q_{/k}) \arrow[r] & H^0_{cdh}(X,\Omega^q_{/F}) \arrow[r] & 0
\end{tikzcd}$$

The third map in the bottom row is onto by (4.6). The first vertical map is an isomorphism by (4.7). By (4.4a), the second is an isomorphism if $q\le n$ and onto if $q=n+1$. It follows that the same is true of the third vertical map. □

Proposition 4.8. *Assume $n\ge 0$. If $HH(/k)$ satisfies n-cdh-descent on R, then so does $HH(/F)$.*

Proof. By Lemma 2.3, the hypothesis is equivalent to saying that the following conditions hold simultaneously.

$$HH^{(q)}_m(R/k)=0 \quad \text{if } 0\le q<m\le n \tag{4.8a}$$

$$\Omega^q_{R/k}\to H^0_{cdh}(X,\Omega^q_{/k}) \quad \text{is bijective if } q\le n \text{ and onto if } q=n+1 \tag{4.8b}$$

$$H^p_{cdh}(X,\Omega^q_{/k})=0 \quad \text{if } p>0 \text{ and } q\le p+n+1. \tag{4.8c}$$

We have to prove that the following conditions hold

$$HH^{(q)}_m(R/F)=0 \quad \text{if } 0\le q<m\le n \tag{4.9a}$$

$$\Omega^q_{R/F}\to H^0_{cdh}(X,\Omega^q_{/F}) \quad \text{is bijective if } q\le n \text{ and onto if } q=n+1 \tag{4.9b}$$

$$H^p_{cdh}(X,\Omega^q_{/F})=0 \quad \text{if } p>0 \text{ and } q\le p+n+1. \tag{4.9c}$$

Using (4.8c), the spectral sequence of 4.2, and induction, we obtain (4.9c). Hence (4.8b) implies (4.9b), by Lemma 4.4. To prove (4.9a) we proceed by induction on q. The case $q=0$ is just the fact that $HH^{(0)}_m(A/k)=0$ for any $m>0$, any field k and any k-algebra A. Assume $n\ge q\ge 1$, and that we have $HH^{(q')}_m(R/F)=0$ for all $m\le n$ and all $q'<\min\{m,q\}$. By (4.9b) and (4.5), the spectral sequence of Lemma 4.1 collapses for $j=0$ to yield

$${}_qE^\infty_{i,-i}={}_qE^1_{i,-i}.$$

Given this, (4.9a) follows from (4.8a) by induction. □

Lemma 4.10. *Let F be a field, and R a local F-algebra essentially of finite type. Then there exists a field $F\subset E\subset R$ such that R is isomorphic to the localization of a finite type algebra over E at a maximal ideal.*

Proof. The hypothesis on R means that there exist an F-algebra A of finite type and a prime ideal $P \subset A$ such that $R = A_P$. Suppose that $\dim(A) = k+h$ and $ht(P) = h$. By Noether normalization, there is a polynomial subring $S = F[t_1, \dots, t_k]$ of A meeting P in 0, and the field $R/P = A_P/PA_P$ is a finite extension of $E = F(t_1, \dots, t_k)$. There is an evident inclusion of E in R, and R is the localization of the finite type E-algebra $A \otimes_S E$ at a maximal ideal. □

The results of this section, together with Theorem 3.1, allow us to prove the following global regularity criterion.

Theorem 4.11. *Let $k \subseteq F$ be fields of characteristic 0, and let R be an F-algebra essentially of finite type.*
If $HH(/k)$ satisfies h-cdh-descent on R, then R is smooth in codimension h. That is, for every prime ideal P of height h, the local ring R_P is regular.

Proof. Let P be a prime ideal of R of height h. Since $HH(/F)$ satisfies *h-cdh* descent on R by Proposition 4.8, it also satisfies *h-cdh* descent on the localization R_P, by Corollary 2.9. By Lemma 4.10, there is a field $F \subset E \subset R_P$ such that R_P is the localization at a maximal ideal of an algebra of finite type over E. By 4.8, $HH(/E)$ satisfies *h-cdh* descent on R_P. Because $\dim(R_P) = h$, Theorem 3.1 implies that R_P is smooth over E, and hence is regular. □

Once again, let F be a field of characteristic 0.

Theorem 4.12. *Suppose R is an F-algebra essentially of finite type. If R is K_{h+1}-regular for some $h \geq 0$, then R is regular in codimension h. In particular, if R is $K_{\dim(R)+1}$-regular, then R is regular, and hence smooth over F.*

Proof. If R is K_{h+1}-regular, then $HH(/\mathbb{Q})$ satisfies h-descent on R, by Corollary 1.7. The assertion now follows from Theorem 4.11. □

5. A nonreduced scheme which is K-regular

This section is devoted to the counterexample stated in Theorem 0.2, which reappears here as Theorem 5.2.

Lemma 5.1. *Let X be a smooth projective elliptic curve over a field F with basepoint Q, and let L be a degree zero line bundle L on X. Assume that L is not an element of odd order in the Picard group. Then $H^*(X, L^{\otimes 2n+1}) = 0$ for all $n \in \mathbb{Z}$.*

Proof. Because $\mathcal{O}_X$ is a dualizing sheaf, we are reduced by Serre duality to proving that $H^0(X, L^{2n+1}) = 0$. Because X is elliptic, there exists a rational point $P \in X$ such that $L := L(P - Q)$. Now if $H^0(X, L^{\otimes 2n+1})$ were nonzero, there would exist an element f in the function field of X with $\mathrm{div}(f) = (2n+1)(P-Q)$. But because L is not an odd torsion element, there is no such f. □

Theorem 5.2. *Let X be a smooth projective elliptic curve over a field F of characteristic 0, and let L be as in Lemma 5.1. Write Y for the nonreduced scheme with the same underlying space as X but with structure sheaf $\mathcal{O}_Y = \mathcal{O}_X \oplus L$, where L is regarded as a square-zero ideal.*

Then for all n, $K_n(Y) = K_n(X)$ and Y is K_n-regular.

Proof. As X is regular, and hence K_n-regular, it suffices to show that $K(Y\times\mathbb{A}^m)\to K(X\times\mathbb{A}^m)\cong K(X)$ is an equivalence for all $m\geq 0$. We shall prove the equivalent assertion that the relative homotopy groups $K(Y\times\mathbb{A}^m, X\times\mathbb{A}^m)$ are zero. By Goodwillie's theorem [6] and Zariski descent, these relative K-groups are isomorphic to the corresponding relative cyclic homology groups over $\mathbb{Q}$. By base-change (see [8]) it suffices to show that the relative groups $HH_n^{rel}=HH_n(Y,X)$ vanish for all n. By Zariski descent, it suffices to show that $H^0(X,HH_n^{rel})$ and $H^1(X,HH_n^{rel})$ vanish for all n. From Lemma 5.3 below and the fact that $\Omega^1_{X/F}\cong\mathcal{O}_X$, we see that the Zariski sheaves HH_n^{rel} are sums of odd tensor powers of L when F is a number field, and odd tensor powers of L tensored over F with vector spaces $\Omega^i_{F/\mathbb{Q}}$ in general. But the cohomology of such powers vanishes by Lemma 5.1. □

The following lemma is well-known, at least in the case when L is free. We include a proof for the sake of completeness. For simplicity, we write $HH_*(R)$ for $HH_*(R/k)$.

Lemma 5.3. *Let k be a field with $char(k)\neq 2$, R a commutative k-algebra, and L a projective R-module of rank 1. Put $A=R\oplus L$. Let M_* denote the graded R-module*

$$M_p=\begin{cases} L^{\otimes_R(p+1)} & p\geq 0 \text{ even},\\ L^{\otimes_R p} & p>0 \text{ odd}.\end{cases}$$

Then for relative Hochschild homology over k,

$$HH_n(A,L)=\bigoplus_{p+q=n} M_p\otimes_R HH_q(R).$$

Proof. Let $C_*(A,L)$ be the relative Hochschild complex; the subspace $L^{\otimes_k 2m+1}$ of $C_{2m}(A,L)$ consists of cycles, and induces a map $M_{2m}=L^{\otimes_R 2m+1}\to HH_{2m}(A,L)$, because for $x_i\in L$ and $r\in R$ we have

$$(-1)^i b(x_0\otimes\cdots x_i\otimes r\otimes x_{i+1}\cdots)=(x_0\otimes\cdots x_i r\otimes x_{i+1}\cdots)-(x_0\otimes\cdots x_i\otimes r x_{i+1}\cdots).$$

Because $Bb+bB=0$, where $B:C_*(A,L)\to C_{*+1}(A,L)$ is the Connes operator, the subspace $B(L^{\otimes_k 2m+1})$ of $C_{2m+1}(A,L)$ also consists of cycles and induces a map $M_{2m+1}=L^{\otimes_R 2m+1}\to HH_{2m}(A,L)$. Thus we have a graded map $M_*\to HH_{2m}(A,L)$. Because $HH_*(A,L)$ is a graded module over $HH_*(R)$, we get a canonical R-module map from $M_*\otimes_R HH_*(R)$ to $HH_*(A,L)$. To see that it is an isomorphism, we may assume R is local, whence $A=R[x]/<x^2>$. By the Künneth formula, we are reduced to the case $R=k$, which is straightforward. □

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